3701. Show that, for values of x close to zero, the second derivative of the function $\mathrm{f}(x) = x \sin x \cos 2x$ may be approximated by $\mathrm{f}''(x) \approx 2 - 24x^2$.

3702. Parametric curves S and L are defined as follows, for parameters $p, q \in [0, \infty)$:

- S is a spiral with equations

$$\begin{aligned} x &= p \cos p, \\ y &= p \sin p. \end{aligned}$$

- L is a half-line with vector equation $\mathbf{r} = \mathbf{a}q$ for some constant non-zero vector $\mathbf{a}$.

(a) On the same set of axes, sketch S and L.

(b) Prove that the points of intersection form two arithmetic progressions in the parameters p and q.

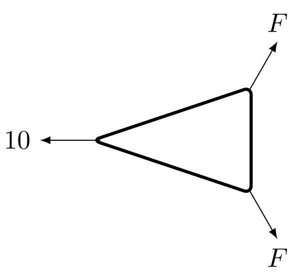
3703. Find $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$.

3704. In a transforming triangle ABC, the lengths a and b of two sides are fixed, while the angle between them increases constantly by 1 radian per second. The length c of the third side duly changes. Prove that the greatest rate of change of c^2 is $2ab$.

3705. In a particular sample, $\sum x^2 < \sum x$. Give, using set notation, all possible values of $\bar{x}$.

3706. Find the equations of the horizontal asymptotes of the curve $4xy - xy^2 + 3 = 0$.

3707. A loop of smooth, light string is pulled taut, in equilibrium, by three coplanar forces, magnitudes $10, F$ and F N. It forms an isosceles triangle with side lengths $\left(2, \sqrt{5}+1, \sqrt{5}+1\right)$:



(a) Explain why the lines of action of the forces are the angle bisectors of the triangle formed by the loop of string.

(b) Find F, to 3sf.

3708. Evaluate $\displaystyle\sum_{i=1}^{10}\left(\sum_{j=1}^{10} ij\right)$.

3709. Point M is defined on the parabola $y = x^2$, with coordinates $\left(m, m^2\right)$. A tangent line is drawn at M. Show that, if this tangent passes through the point (a, b), then

$$m = a \pm \sqrt{a^2 - b}.$$

3710. The variable Z has the distribution $Z \sim N\left(\mu, \sigma^2\right)$. Show that $\mathbb{P}(Z - \mu < \sigma \mid Z - \mu < 2\sigma) \approx 86\%$.

3711. A function g is defined as $\mathrm{g}(x) = \dfrac{x}{1-x}$.

(a) Find a simplified expression for $\mathrm{g}^n(x)$.

(b) Give the largest real domain over which g^n can be defined.

3712. Parametric equations are given as

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}.$$

Prove that these describe the unit circle.

3713. Show that the rate of change of $\tan x$ with respect to $\cot x$ is $-\tan^2 x$.

3714. Show that one of the following two curves has no intersections with the parabola $y = x^2$:

① $y = \dfrac{1}{1-x^2}$,

② $y = \dfrac{1}{x^2-1}$.

3715. Take g to be 10 in this question.

A bouncy ball is dropped from 20 metres up, then another is dropped 1 second later. Assuming that, in bouncing on level ground, the first ball loses no speed, find the time that elapses before the balls collide.

3716. Express $\left(x^3 + 3x^2\right) \div \left(x + 2\right)$ as the sum of a quadratic in x and an algebraic fraction with constant numerator.

3717. A differential equation is given as

$$\mathrm{f}'(x) = \ln 2 \times \mathrm{f}(x).$$

Solve the DE, answering in fully simplified form.

3718. Evaluate $\displaystyle\lim_{p,q \to x} \frac{p^5 - q^5}{p - q}$.

3719. Functions f and g are defined over $\mathbb{R}$, with ranges $[a, b]$ and $[c, d]$ respectively, where $a < b < c < d$. For each of the following, give the smallest set which can be guaranteed to contain the range of the function h:

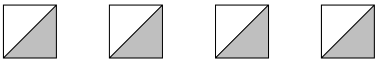
(a) $\mathrm{h} : x \mapsto \mathrm{f}(x) + \mathrm{g}(x)$,

(b) $\mathrm{h} : x \mapsto \mathrm{f}(x) - \mathrm{g}(x)$.

3720. You are given that the curves $\log_2 x + \log_4 y = 1$ and $x + y = k$ intersect exactly once.

(a) Sketch the curves.
(b) Determine the value of k.

3721. The four tiles below are placed together, in random orientations, to form a two-by-two square.



Find the probability that the resulting shape has rotational symmetry order 2 (but not order 4).

3722. "There are polynomial functions $x \mapsto \mathrm{f}(x)$ which are convex and increasing for $x \in (k-1, k)$, and also stationary at $x = k$."

True or false?

3723. You are given that the parabolae $y = \frac{1}{4}x^2 + p$ and $x = \frac{1}{4}y^2 + p$ do not intersect. Find all possible values of the constant p.

3724. Over the largest real domain, find the range of

$$\mathrm{f}(x) = \frac{x^2 + 8x + 9}{x^2 + 8x - 9}.$$

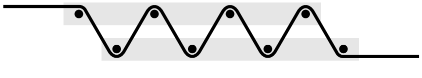
3725. The cubic $y = x^3 - x$ is drawn, with a tangent at $x = a$. The tangent crosses the curve at $x = a - 6$. Determine the value of a.

3726. Two unit spheres have the following equations:

$$S_1 : x^2 + y^2 + z^2 = 1,$$
$$S_2 : (x-1)^2 + y^2 + z^2 = 1.$$

Show the set of (x, y, z) points in common to S_1 and S_2 is a circle, giving the centre and radius.

3727. In a loom, a thread winds, under tension T, around $2n$ small, smooth pegs (black) set in two banks (grey), as shown. The straight sections of thread are parallel to the edges of an equilateral triangle.



Show that each bank experiences a total force of

(a) $\frac{1}{2}T$ parallel to its length,
(b) $\left(n - \frac{1}{2}\right)\sqrt{3}\,T$ perpendicular to its length.

3728. The graph $\mathrm{f}(x) + \mathrm{f}(y) = 1$, where f is a function, is stretched by scale factor 3 in the y direction. Write down the equation of the new graph.

3729. Find the linear small-angle approximation for the trigonometric expression $\cot(\theta - \frac{\pi}{2})$.

3730. Find the lengths of the edges of a cuboid whose face diagonals have lengths $500, 707, 843$.

3731. Point $A : \left(a, a^2\right)$ is drawn on the parabola $y = x^2$. A perpendicular is dropped from A to the x axis at point P. A tangent is drawn to $y = x^2$ at A, which crosses the x axis at X and the y axis at Y.

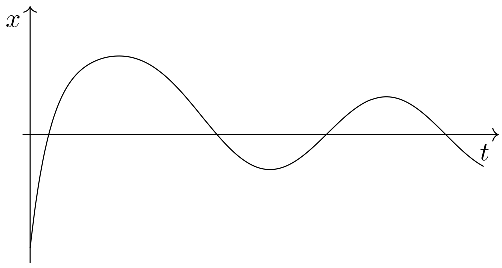
(a) Sketch the scenario.
(b) Prove that $\triangle OXY$ and $\triangle APX$ are congruent.

3732. Prove that the derivative of $\ln x$ is x^{-1}.

3733. The displacement x of a moving part of a machine, t seconds after the machine is jolted, is modelled with the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 2\sin t.$$

The solution $y = (7t + 2)e^{-t} - \cos t$ is proposed. This is shown below, for $0 \leq t \leq 12$:



(a) Verify that this curve satisfies the DE.
(b) Using the Newton-Raphson method, find the time at which greatest positive displacement occurs.
(c) Describe the long-term behaviour.

3734. In each case, find the limit in terms of x:

(a) $\displaystyle\lim_{a \to x} \frac{a - x}{a^2 - x^2}$,
(b) $\displaystyle\lim_{a \to x} \frac{a - x}{a^3 - x^3}$.

3735. For a function $x \mapsto \mathrm{f}(x)$, the function $x \mapsto \mathrm{F}(x)$ is defined as

$$\mathrm{F}(x) = \int_0^x \mathrm{f}(t)\,dt.$$

You are given that $\mathrm{F}(a) = p$ and $\mathrm{F}(b) = q$.

Evaluate the following, giving answers in terms of p and q:

(a) $\displaystyle\int_a^b \mathrm{f}(x)\,dx$,
(b) $\displaystyle\int_{a+1}^{b+1} \mathrm{f}(x-1)\,dx$,
(c) $\displaystyle\int_{3a}^{3b} \mathrm{f}\left(\tfrac{1}{3}x\right)\,dx$.

3736. Show that $\sqrt{x+2y}+\sqrt{x-2y}=2$ is parabolic.

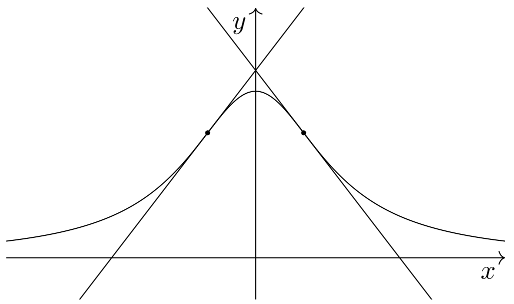
3737. Use a double-angle formula to show that

$$\int_0^{2\pi} \sqrt{2-2\cos t}\, dt = 8.$$

3738. The diagram below shows

$$y=\frac{8}{1+x^2}.$$

Two tangents have been drawn. Each crosses the curve, intersecting it at exactly one point.



Show that the tangents intersect at $y=9$.

3739. The *logarithmic spiral* has parametric equations as follows, with a and k positive constants:

$$x=ae^{kt}\cos t,$$
$$y=ae^{kt}\sin t.$$

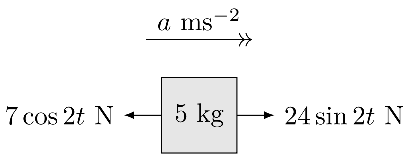
(a) Sketch the curve.

(b) Show that the points at which the gradient is 1 have parameter t satisfying

$$\tan t=\frac{k-1}{k+1}.$$

3740. A sample of size 100 is taken, which has mean 63.2 and standard deviation 8.1. A set of 20 data, whose mean and standard deviation are 64.8 and 7.6, is then removed from the sample. Calculate the new mean and standard deviation.

3741. Two oscillating forces are applied to an object, as modelled in the following force diagram:



Find the maximum value of the magnitude of the acceleration during the motion.

3742. The expansion of $(x+1)^m(x-1)^n$, for $m,n\in\mathbb{N}$, begins $x^7-3x^6+\ldots$. Find m and n.

3743. The curve $x-2y+\sqrt{x+y}=1$ has an endpoint.

(a) Find this endpoint.

(b) By differentiating implicitly, show that

$$\frac{dy}{dx}=\frac{2\sqrt{x+y}+1}{4\sqrt{x+y}-1}.$$

(c) Hence, show that the line $x+y=0$ is tangent to the curve at its end-point.

3744. Ships A and B are both travelling at speed u. At time $t=0$, ship A is a distance d_0 due north of B. Ship A travels eastward, ship B north-eastward.

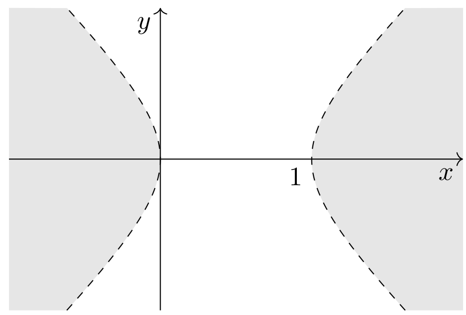
(a) Find an expression for d^2, the square of the distance between the ships, at time t.

(b) Show that closest approach occurs at

$$t=\frac{(1+\sqrt{2})d_0}{2u}.$$

3745. A geometric sequence has $u_1=\sin x$, $u_2=\sin 2x$. Solve the equation $u_3=0$, for $x\in[0,2\pi)$.

3746. On the graph, the points satisfying the inequality $x^2-x-y^2>0$ are shaded:



It is given that none of the points shaded above satisfy $x^2-2rx+y^2=0$. Determine the greatest possible value of r.

3747. Events A and B have probabilities $\mathbb{P}(A)=0.5$ and $\mathbb{P}(B)=0.75$. Determine all possible values of $\mathbb{P}(A\mid B)$, giving your answer in set notation.

3748. In this question, do not use a calculator.

Solve the equation $3x^{\frac{1}{3}}+11=4x^{-\frac{1}{3}}$.

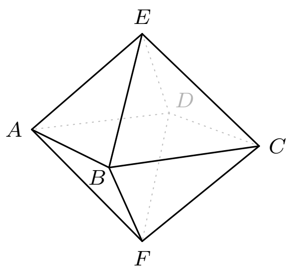
3749. A graph, whose equation is $\mathrm{f}(x)+\mathrm{g}(y)=10$ for some functions f and g, is translated by $2\mathbf{i}-3\mathbf{j}$. Write down the equation of the transformed graph.

3750. A function is defined over $\mathbb{R}^+$ as

$$\mathrm{g}: x\mapsto \left(x-\sqrt{x}\right)\left(1+\sqrt{x}\right).$$

Show that the range is $\left\{y\in\mathbb{R}: y\geq -\frac{2\sqrt{3}}{9}\right\}$.

3751. A regular octahedron is shown below. An ant is walking the surface of the octahedron, starting at A. It chooses an edge at random and walks its length. Upon reaching another vertex, it chooses a new edge at random, never leaving the edges and never travelling the same edge twice.



Prove that, when the ant has walked the length of all twelve edges, it must be back at A.

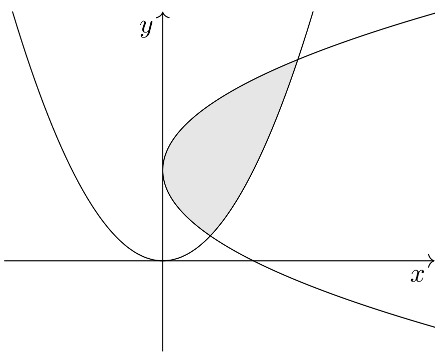
3752. You are given that the quartic $ax^4 + bx^2 + c = 0$ has exactly two real roots. State, with a reason, whether the following equations can be guaranteed to have exactly two real roots:

(a) $ax^2 + bx + c = 0$,
(b) $ax^6 + bx^3 + c = 0$,
(c) $ax^8 + bx^4 + c = 0$.

3753. Line L, with equation $y = x \tan\alpha$, is reflected in the mirror line $y = x\tan\beta$, for constant angles α and β. Find the equation of the image of L under this reflection.

3754. A circle is drawn, with radius $r > 0$ and centre $(0, r)$, such that the circle intersects with the curve $y = x^2$ exactly once. Determine all possible values for r, giving your answer in set notation.

3755. Find, to 3sf, the area of the shaded region in the following diagram. The parabolae have equations $y = x^2$ and $x = (y-1)^2$.



3756. *L'Hôpital's rule* states that limits such as the one below can be found by differentiating numerator and denominator before taking the limit. Using L'Hôpital's rule, show that

$$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x} = 6.$$

3757. A curve is given, for non-zero constants a, b, by

$$ax + by = (bx - ay)^2.$$

Show that, at the origin, the normal to this curve has equation $ax + by = 0$.

3758. Variables x_i, for $i = 1, 2, 3$, satisfy the equations

$$\sum_{i=1}^{3} x_i = 3, \qquad \sum_{i=1}^{3} x_i^2 = 5$$

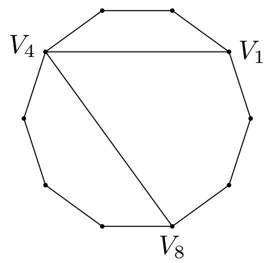
(a) Multiply out $\left(\sum_{i=1}^{3} x_i\right)^2$
(b) Hence, evaluate $x_1x_2 + x_2x_3 + x_3x_1$.

3759. A student proposes partial fractions in the form

$$\frac{1}{x^2(x-6)} \equiv \frac{A}{x^2} + \frac{B}{x-6}.$$

Explain why this will not work, and write down the correct form.

3760. A regular decagon has vertices $V_1, V_2, V_3, ..., V_{10}$, labelled in order around the perimeter.



Find angle $V_1V_4V_8$.

3761. A quintic curve Q has equation

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{6}x^4 + \tfrac{1}{20}x^5.$$

Show that Q has exactly one point of inflection.

3762. A computer generates random numbers $X_1, X_2, ...$ according to a normal distribution $N(0, 1)$. Find the following probabilities:

(a) $\mathbb{P}(X_1 > 1)$,
(b) $\mathbb{P}(X_1 + X_2 + X_3 > 1)$.

3763. The quadratic approximation to $y = \cos x$, for x values close to zero, is

$$y = 1 - \tfrac{1}{2}x^2.$$

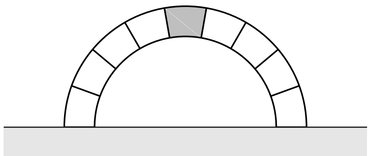
By considering a transformation of the above, find the quadratic approximation to $y = \cos x$ for x values close to π.

3764. Assuming the relevant compound-angle formula, prove that, for $x, y \in (-1, 1)$,

$$\arctan x + \arctan y \equiv \arctan\left(\frac{x+y}{1-xy}\right).$$

3765. Three red and three blue counters have been placed in a row, in a random order. Given that the red counters have ended up in a group, find the probability that the blue counters have too.

3766. Bridges and arches are often built with a *keystone*, which sits at the centre. Depicted is a circular arch with nine stones, with the keystone shaded.



Consider an arch of $2k+1$ symmetrical stones, each of mass m, for $k \in \mathbb{N}$. Neglecting friction, show that the contact force between the keystone and each of its neighbouring stones is

$$R = \tfrac{1}{2}mg \operatorname{cosec}\left(\frac{90°}{2k+1}\right).$$

3767. Sketch $y = \sqrt{\cos x}$.

3768. One of the following statements is true; the other is not. Prove the one and disprove the other.

(a) $\displaystyle\lim_{x\to 0}\frac{\mathrm{f}(x)}{\mathrm{g}(x)} = 0 \implies \lim_{x\to 0}\mathrm{f}(x) = 0,$

(b) $\displaystyle\frac{\mathrm{f}(x)}{\mathrm{g}(x)} = 0 \implies \mathrm{f}(x) = 0.$

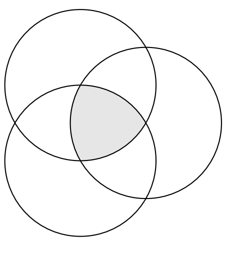
3769. Two functions f and g are monic quadratics. They each have two distinct roots. at $\{a, b\}$ and $\{b, c\}$ respectively, where a, b, c are in AP.

Sketch the following graphs:

(a) $y = \mathrm{f}(x) + \mathrm{g}(x)$,

(b) $y = \mathrm{f}(x)\,\mathrm{g}(x)$.

3770. Three unit circles are drawn on a plane, such that each centre is at an intersection.



Show that the area of the region common to all three circles is $\frac{1}{2}\left(\pi - \sqrt{3}\right)$.

3771. This question concerns the differentiation of x^4 from first principles.

(a) Factorise $x^4 - p^4$ fully.

(b) Hence, show that $\displaystyle\lim_{p\to x}\frac{x^4 - p^4}{x - p} \equiv 4x^3$.

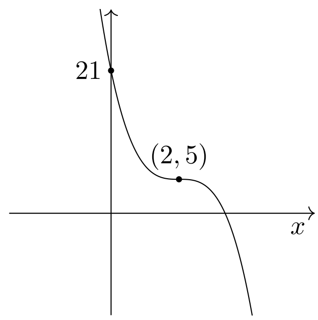
3772. Find the acute angle between the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$, giving your answer as $\arctan a$.

3773. By calculating derivatives, or otherwise, show that $y = \sin x$ and $x = \sin 2y$, both defined in radians, have exactly three points of intersection. You do not need to find the coordinates of the points of intersection.

3774. Verify that $\mathrm{f}(x) = x\ln x$ satisfies the differential equation

$$\frac{\mathrm{f}'(x)}{\mathrm{f}''(x)} = \mathrm{f}(x) + x.$$

3775. A cubic graph with equation $y = ax^3 + bx^2 + cx + d$ passes through $(0, 21)$ and has a stationary point of inflection at $(2, 5)$.



Find the coefficients a, b, c, d.

3776. True or false?

(a) $x^2 = 0 \implies x^3 = 0$,

(b) $x^3 = 0 \implies x^4 = 0$,

(c) $x^k = 0 \implies x^{k+1} = 0$.

3777. Show carefully that the following iteration has no values x_0 for which it is periodic with period 2:

$$x_{n+1} = \frac{1}{x_n + 1}.$$

3778. A family of ellipses is defined by

$$(x-a)^2 + (ay-1)^2 = 1.$$

Sketch the locus of the centres of the ellipses.

3779. By rearranging and factorising, solve the following equation, for $x \in [0, 2\pi)$:

$$6\sin x + \sqrt{3} = 2\sqrt{3}\cos x + 3\tan x.$$

3780. In a game, three dice are rolled. Any die showing an odd number is then removed. Any remaining dice are rolled again. The game ends when all of the dice have been removed. Find the probability that the game ends after

(a) one roll,

(b) two rolls.

3781. Real variables $x, y > 0$ satisfy the equations

$$6x^6 + x^4y^4 - x^2y^8 = 0,$$
$$x^2 + y^4 = 1.$$

Determine all possible values of x and y.

3782. A *rational function* is defined, for polynomials $\mathrm{p}(x)$ and $\mathrm{q}(x)$ which share no common factor, as

$$\mathrm{f}(x) = \frac{\mathrm{p}(x)}{\mathrm{q}(x)}.$$

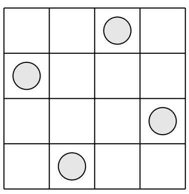
(a) Explain why, if $\mathrm{q}(x)$ has a factor of $(x-\beta)^n$, for $n \in \mathbb{N}$, then $\mathrm{f}(x)$ has an asymptote at $x = \beta$.

(b) Determine the condition on the index n for $\mathrm{f}(x)$ to change sign at $x = \beta$.

(c) Hence, sketch the graph

$$y = \frac{1}{(x-2)^4(x+2)^4}.$$

3783. On a four-by-four grid, four identical counters are placed at random, on distinct squares.



Find the probability that, as in the example, no two counters occupy the same row or column.

3784. Solve $2\sin^2 x + \cos|x| = 1$ for $x \in [-\pi, \pi]$.

3785. You are given that the equations $y = x^2 + a$ and $y = bx^2$ have no simultaneous solutions. On a set of Cartesian (a, b) axes, sketch regions describing the sets of possible values of the constants a, b.

3786. Prove that $\sec\left(\frac{3\pi}{2} - \theta\right) \equiv -\operatorname{cosec}\theta$.

3787. Determine the positive value of a such that

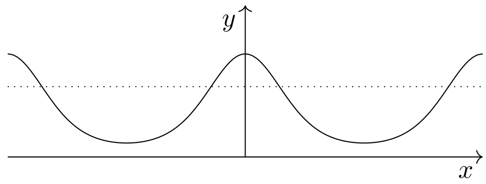
$$\int_0^1 \frac{a}{(x+a)(x+2a)}\,dx = \ln\tfrac{8}{7}.$$

3788. Prove that, if $x, y \in \mathbb{R}$ satisfy $x^2 + y^2 = 1$, then the maximum value of $x^3 + y^3$ is 1.

3789. A projectile is launched from an origin, at speed u, at an angle θ above the horizontal. Show that the equation of the trajectory can be written as

$$y + \frac{gx^2}{2u^2}\tan^2\theta - x\tan\theta + \frac{gx^2}{2u^2} = 0.$$

3790. A curve is shown, with equation $y = e^{\cos x}$. It has infinitely many points of inflection, all of which lie on the line $y = k$, shown dotted below:



Show that $\ln k = \phi$, where $\phi = \dfrac{-1+\sqrt{5}}{2}$.

3791. A polynomial h is convex everywhere. Show that the range of h is $[k, \infty)$, for some $k \in \mathbb{R}$.

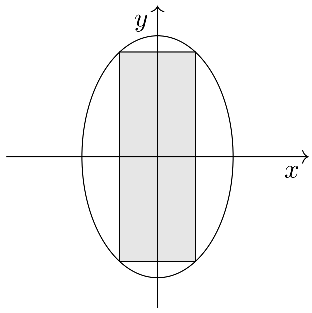
3792. Evaluate $\displaystyle\sum_{i=1}^{\infty} 2^{2i} \cdot 5^{1-i}$.

3793. Functions f and g are both defined over $\mathbb{R}$. Their ranges are given as $[-1, 1]$ and $[0, \infty)$ respectively. Either prove or disprove the following statement: "The range of the function $x \mapsto \mathrm{f}(x) + \mathrm{g}(x)$ includes all real numbers greater than 1."

3794. The diagram below shows the ellipse

$$4x^2 + y^2 = 4.$$

A rectangle of area A is drawn inside the ellipse, with sides parallel to the x and y axes.



Show that $A \leq 4$.

3795. *Descartes' theorem* says that, if a fourth circle is placed in the gap between three others, such that all four are pairwise tangent, the radii r_i satisfy

$$\left(\sum_{i=1}^{4} \frac{1}{r_i}\right)^2 = 2\sum_{i=1}^{4} \frac{1}{r_i^2}.$$

Solve to find the last radius with radii $2, 4, 4$.

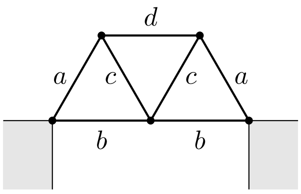
3796. A monic quartic curve $y = \mathrm{f}(x)$ has a double root at the origin, and is convex on $(-\infty, -2)$, concave on $(-2, 3)$ and convex on $(3, \infty)$.

(a) Show that, for a real constant k, the second derivative may be expressed as

$$\mathrm{f}''(x) = k(x^2 - x - 6).$$

(b) Express $\mathrm{f}(x)$ in polynomial form.

3797. A bridge over a river is built of seven iron girders.



State, without any calculation, whether the girders marked a, b, c, d are in tension or compression.

3798. In a data-generating process P_n, n fair coins are tossed, and the number of heads counted. Find the probability that P_{n+1} produces more heads than P_n, in the following cases:

(a) $n = 1$,

(b) $n = 2$.

3799. Four statements are given below, which refer to a function f and to variables u and x. You are given that $u = 3x + 2$. Which of the statements are definitely true?

(a) $\mathrm{f}'(x) = \mathrm{f}'(u)$,

(b) $\frac{d}{dx}\,\mathrm{f}(x) = \frac{d}{du}\,\mathrm{f}(u)$,

(c) $\frac{d}{dx}\,\mathrm{f}(3x+2) = \frac{d}{du}\,\mathrm{f}(u)$,

(d) $\frac{d}{dx}\,\mathrm{f}(3x+2) = \frac{d}{dx}\,\mathrm{f}(u)$.

3800. Show that the tangent to $y = x(\sin x + \cos x)$ at the origin intersects the curve again at $x = \frac{\pi}{2}$.

——— End of 38th Hundred ———